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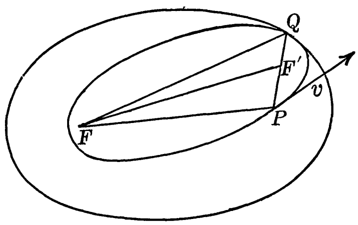
328 (Mechanics). Proposed by J. B. REYNOLDS, Lehigh University.

Find the envelope of all possible trajectories when a particle is projected with a constant velocity, v, from a fixed point at a distance a from the center of attraction under the law of gravitation.

I. Solution by the Proposer.

For parabolic or hyperbolic velocities there is evidently no envelope for all directions of projection as any point in space may be reached. For velocities giving elliptic orbits we have



$$v^2 = \mu\left(\frac{2}{a} - \frac{1}{A}\right) \tag{1}$$

in which A is the semi-major axis of the orbit; whence

$$A = \frac{\mu a}{2\mu - av^2} \tag{2}$$

In the figure, let P be the point of projection, F and F' the foci of the path and Q the point where PF' extended cuts the orbit. We will prove that the locus of Q is an ellipse.

Now

$$\begin{aligned} FQ + F'Q &= 2A \\ a \quad + F'P &= 2A. \end{aligned} \tag{1}$$

Adding, we get

$$FQ + F'Q + F'P = 4A - a = \frac{2\mu a + v^2a^2}{2\mu - av^2} = a \cdot \frac{2\mu + av^2}{2\mu - av^2}. \tag{2}$$

or

$$FQ + PQ = \text{const.} \tag{3}$$

Hence, the locus of Q is an ellipse. Furthermore, at Q both ellipses have the same normal so that the inner ellipse does not cross the outer. The ellipsoid of revolution generated by revolving the outer ellipse about FP is the envelope of all the trajectories in question.

If the major axis of the generating ellipse is $2a'$ and its eccentricity e', we have by (2)

$$2a' = a\frac{2\mu + av^2}{2\mu - av^2}. \tag{4}$$

Suppose the particle projected directly away from F. Then since $vdv/dr = -\mu/r^2$, we have, when the particle comes to rest at distance r from F,

$$-v^2 = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right) \quad \text{or} \quad r = \frac{2\mu a}{2\mu - av^2}.$$

This value of r is evidently $a'(1 + e')$; so that

$$a'(1 + e') = \frac{2\mu a'}{2\mu - av^2} \cdots. \tag{5}$$

Now (4) and (5) give

$$e' = \frac{2\mu - av^2}{2\mu + av^2},$$

and the envelope is consequently determined.

II. Solution by C. F. Gummer, Kingston, Ontario.

The origin being the center of attraction, $(a, 0)$ the initial position, and μ/ρ^2 the attraction, the equations of motion will be

(1) $$\ddot{\rho} - \rho\dot{\theta}^2 = -\mu\rho^{-2},$$

and

(2) $$2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} = 0.$$

By (2), $\rho^2\dot{\theta} = A$, so that (1) gives

$$\ddot{\rho} - A^2\rho^{-3} = -\mu\rho^{-2},$$

or

$$\dot{\rho} = \sqrt{-A^2\rho^{-2} + 2\mu\rho^{-1} + B}.$$

The speed is

$$\sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2} = \sqrt{2\mu\rho^{-1} + B}.$$

Hence,

$$B = v^2 - 2\mu a^{-1}.$$

Hence,

$$\frac{d\rho}{d\theta} = \frac{\dot{\rho}}{\dot{\theta}} = \rho^2 A^{-1}\sqrt{v^2 - 2\mu a^{-1} + 2\mu\rho^{-1} - A^2\rho^{-2}}$$

and therefore,

$$\theta = -\int_{a^{-1}}^{\rho^{-1}} A^{-1}\sqrt{v^2 - 2\mu a^{-1} + 2\mu z - A^2z^2}\, dz;$$

that is,

$$\rho^{-1} = \mu A^{-2} - (\mu A^{-2} - a^{-1})\cos\theta + \sqrt{v^2A^{-2} - a^{-2}}\cdot\sin\theta,$$

which is the equation of the trajectory.

Writing this in the form,

$$\mu^2a^2(1-\cos\theta)^2+\{2\mu a(1-\cos\theta)(\cos\theta-a\rho^{-1})-a^2v^2\sin^2\theta\}A^2+\{(\cos\theta-a\rho^{-1})^2+\sin^2\theta\}A^4=0,$$

and applying the condition for equal roots in A^2, we get for the envelope

$$\frac{4\mu a^2v^2}{4\mu^2 - a^2v^4}\frac{1}{\rho} = 1 - \frac{2\mu - av^2}{2\mu + av^2}\cos\theta,$$

the equation of an ellipse.

349 (Mechanics). Proposed by S. A. COREY, Albia, Iowa.

A 9-pound weight is attached to a string which passes over a smooth fixed pulley. The other end of the string is fastened to and supports a smooth pulley P_1 of weight 1 pound over which passes a second string, one end attached to a 3-pound weight and the other end attached to and supporting another smooth pulley P_2 of weight 1 pound. Over the pulley P_2 passes a third string supporting weights 2 pounds and $3\frac{1}{3}$ pounds.

If the system is acted upon by gravity alone show that the acceleration of the 9-pound weight, $3\frac{1}{3}$-pound weight, and pulley P_2 are 0, $\frac{1}{2}g$, and $\frac{1}{3}g$, respectively.

Determine the motion of the weights when pulleys are not smooth, that is, when friction is present.

II. Solution by William Hoover, Columbus, Ohio.

Calling the fixed pulley P and taking it as the origin of coördinates, we have for the x-coördinate of m_1, x_1; of P_1, $l_1 - x_1$; of m_3, $x_2 + l_1 - x_1$; of P_2, $l_1 + l_2 - (x_1 + x_2)$; of m_4, $x_3 + l_1 + l_2 - (x_1 + x_2)$; of m_5, $l_1 + l_2 + l_3 - (x_1 + x_2 + x_3)$, where m_1, m_3, m_4 and m_5 are the masses of the 9-, 3-, 2-, and $3\frac{1}{3}$-pound weights, respectively.

Under the hypothesis that there is no friction, the equation of motion of system is

$$\begin{aligned} T = {}&\tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}P_1\dot{x}_1^2 + \tfrac{1}{2}m_3(\dot{x}_2 - \dot{x}_1)^2 + \tfrac{1}{2}(\dot{x}_1 + \dot{x}_2)^2 + \tfrac{1}{2}m_4(\dot{x}_3 - \dot{x}_1 - \dot{x}_2)^2 + \tfrac{1}{2}m_5(\dot{x}_1 + \dot{x}_2 \\ &+ \dot{x}_3)^2 = m_1gx_1 + P_1g(l_1 - x_1) + m_3g(x_2 + l_1 - x_1) + P_2g\{l_1 + l_2 - (x_1 + x_2)\} \\ &+ m_4g\{l_1 + l_2 + x_3 - (\dot{x}_1 + x_2)\} + C = V. \end{aligned} \tag{i}$$

Using Lagrange's equations of type

$$\frac{d}{dt}\frac{dT}{d\dot{x}} - \frac{dT}{dx} = \frac{dV}{dx} \tag{ii}$$

there are

$$\begin{aligned} &(a) && \tfrac{58}{3}\ddot{x}_1 + \tfrac{10}{3}\ddot{x}_2 + \tfrac{4}{3}\ddot{x}_3 = g(m_1 - P_1 - m_3 - P_2 - m_4 - m_5) = -\tfrac{4}{3}g, \\ &(b) && \tfrac{10}{3}\ddot{x}_1 + \tfrac{28}{3}\ddot{x}_2 + \tfrac{4}{3}\ddot{x}_3 = g(m_3 - P_2 - m_4 - m_5) = -\tfrac{10}{3}g, \\ &(c) && \tfrac{4}{3}\ddot{x}_1 + \tfrac{4}{3}\ddot{x}_2 + \tfrac{16}{3}\ddot{x}_3 = g(m_4 - m_5) = -\tfrac{4}{3}g, \end{aligned}$$

giving $\ddot{x}_1 = 0$, $\ddot{x}_2 = -\dfrac{g}{3}$, $x_3 = -\dfrac{g}{6}$.